The “Elusive Formulas”^{2}

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Section A – Symbol Table

$\forall$	for all
$\exists$	there exists
$\varnothing$	the empty set
$\in$	is an element of
$\notin$	is not an element of
□ ,□ +	the set of natural numbers
□	the set of integers
□	the set of rational numbers
□	the set of real numbers
□	the set of complex numbers
$\subseteq$	is a subset of
$\vee$	or
$\wedge$	and
$\cap$	union
$\cup$	intersection
$\Rightarrow$	implies
$\Leftrightarrow$, iff	is equivalent to
$\sum_{i=1}^{n} a_i$	$a_1+a_2+a_3+a_4+a_5+...+a_n$
$\prod_{i=1}^{n} a_i$	$a_1 \bullet a_2 \bullet a_3 \bullet a_4 \bullet a_5 \bullet \ldots \bullet a_n$
(a,b) = d	d is the gcd of a and b
[a,b] = d	d is the lcm of a and b

$\tau(a)$	number of factors of a
$\sigma(a)$	sum of the factors of a
$\varphi(a)$	Euler Phi Function
$\mu(a)$	Mobius Function
$\lvert a \rvert$	absolute value of a
$\lfloor a \rfloor$	greatest integer function
$\lceil a \rceil$	least integer function
a:b:c	ratio of a to b to c
a:b:c::d:e:f	ratio of a to b to c=ratio of d to e to f
π	pi $\approx 3.141592653589793\ldots$
e	euler number $\approx 2.718281828459\ldots$
$\log_b(a) = c$	$b^c = a$
$\log(a) = c$	$10^c = a$
n!	$n(n-1)(n-2)(n-3)(n-4)\ldots 3\times 2\times 1$
${}_nP_r$	$\frac{n!}{r!} = n(n-1)(n-2)\ldots(n-r+1)$
${}_nC_r$ or $\binom{n}{r}$	$\dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)(n-2)...(n-r+1)}{n(n-1)(n-2)...(2)(1)}$

$a \equiv b \pmod{c}$ *a* and *b* leave the same remainder when divided by *c*.

Section B – Algebra

- $(a \pm b)^3 = a^3 \pm b^3$ iff a = 0 or b = 0 or $(a \pm b) = 0$
- $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
- $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
- $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = -16s(s - a)(s - b)(s - c)$ when $2s = a+b+c$
- $a^n + b^n = (a + b)(a^{n-1} + b^{n-1}) - ab(a^{n-2} + b^{n-2})$
- $a^n \pm b^n = (a \pm b)(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp a^{n-4}b^3 + \ldots + a^2b^{n-3} \mp ab^{n-2} + b^{n-1})$ [$a^n + b^n$ is only true for odd n.]
- $(a \pm b)^n = {}_nC_0a^n \pm {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 \pm {}_nC_3a^{n-3}b^3 + {}_nC_4a^{n-4}b^4 \pm \ldots \pm {}_nC_{n-2}a^2b^{n-2} + {}_nC_{n-1}ab^{n-1} + {}_nC_nb^n$
- $a(a+1)(a+2)(a+3) = (a^2+3a+1)^2 - 1$

Arithmetic Series: If $a_1, a_2, a_3, \ldots, a_n$ are in arithmetic series with common difference d:	
n^{th} term in terms of m^{th} term	$a_n = a_m + (n - m)d$
Sum of an arithmetic series up to term n	$\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2} = \frac{n(2a_1 + (n-1)d)}{2}$
Geometric Series: If $a_1, a_2, a_3, \ldots, a_n$ are in geometric series with common ratio r:	
n^{th} term of a geometric series	$a_n = a_1r^{n-1}$
Sum of a non-constant ($r \neq 1$) geometric series up to term n	$\sum_{i=1}^{n} a_i = \frac{a_1(1-r^n)}{1-r}$
Sum of an infinite geometric series	$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}$ iff $\lvert r \rvert < 1$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$	$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_1x + a_0 = 0$, a_i is a constant, then	
Sum of roots taken one at a time (the sum of the roots)	$\sum r_i = \frac{-a_{n-1}}{a_n}$
Sum of roots taken two at a time	$\sum_{i \neq j} r_ir_j = \frac{a_{n-2}}{a_n}$
Sum of roots taken p at a time	$\sum_{i \neq j \neq \ldots \neq k} r_ir_j \ldots r_k = (-1)^p \frac{a_{n-p}}{a_n}$
Rational Root Theorem	
If $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_1x + a_0$ is a polynomial with integer coefficients and b/c is a rational root of the equation $P(x) = 0$ (where $(b, c) = 1$), then $b \mid a_0$ and $c \mid a_n$.	

- If P(x) is a polynomial with real coefficients and $P(a + bi) = 0$, then $P(a - bi) = 0$.
- If P(x) is a polynomial with rational coefficients and $P(a + b\sqrt{c}) = 0$, then $P(a - b\sqrt{c}) = 0$.

Section C – Number Theory

- Number Theory mainly concerns □ and □, all variables exist in □ unless stated otherwise

Divisibility: ∀a,b∈□, a≠0: a | b ⇔ ∃k∈□ such that ak = b		
1|a, a|0, a|(±a)	a|b ⇒ a|bc	a|b ∧ b|c ⇒ a|c
a|1 ⇔ a=±1	a|b ∧ a|c ⇒ a|(b±c)	a|bc ∧ (a,b) =1 ⇒ a|c
a|b ∧ b|a ⇔ a=±b	a|b ∧ c|d ⇒ ab|cd	a|c ∧ b|c ∧ (a,b)=1 ⇒ ab|c
Modulo Congruence: ∀a,b,m∈□, m≠0: a ≡ b (mod m) ⇔ m | (a-b)		
Suppose that a≡b (mod m), c≡d (mod m), and p is prime; then:		
a±g ≡ c±g (mod m)	a±b ≡ c±d (mod m)	(g,p)=1 ⇒ g^{p-1} ≡ 1 (mod p)
ag ≡ cg (mod m)	ab ≡ cd (mod m)	(p-1)! ≡ -1 (mod p)
(g,m)=1 ⇒ $g^{\varphi(m)}$ ≡ 1 (mod m)	hf≡hg (mod m) ∧ (m,h)=1 ⇒ f≡g (mod m)	

Fibonacci Sequence

- Sequence of integers beginning with two 1's and each subsequent term is the sum of the previous 2 terms.
- 1,1,2,3,5,8,13,21,34,55,89,144, ...
- F(1)=F(2)=1, for n≥3, F(n)=F(n-1)+F(n-2)
- Let ψ = Golden Ratio = $\frac{\left(\sqrt{5}+1\right)}{2}$, then F(n) = $\frac{\psi^{n}-(-\psi)^{-n}}{\sqrt{5}}$
- F(n)•F(n+3) – F(n+1)•F(n+2) = $(-1)^n$

Farey Series [F_n]

- Ascending sequence of irreducible fractions between 0 and 1 inclusive whose denominator is ≤n
- $F_3 = \frac{0}{1},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{1}{1}$; $F_7 = \frac{0}{1},\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{2}{7},\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{3}{5},\frac{2}{3},\frac{5}{7},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{6}{7},\frac{1}{1}$
- if $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are successive terms in F_n, then bc–ad = de–cf = 1 and $\frac{c}{d} = \frac{a+e}{b+f}$

Number Theory Functions

The following number theory functions have the property that if (a,b)=1, then f(a×b)=f(a)×f(b)

Tau Function: Number of factors of n: $\tau(n) = \prod_{i=1}^{m}\left(1+\alpha_i\right)$

Sigma Function: Sum of factors of n: $\sigma(n) = \prod_{i=1}^{m}\left(\sum_{j=0}^{\alpha_i}\left(p_i^{\,j}\right)\right) = \prod_{i=1}^{m}\left(\frac{p_i^{1+\alpha_i}-1}{p_i-1}\right)$

Euler Phi Function: Number of integers between 0 and n that are relatively prime to n

$$\varphi(n) = \prod_{i=1}^{m}\left(p_i^{\alpha_i}-p_i^{\alpha_i-1}\right) = n□\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)$$

Mobius Function:
$$\mu(n) = \begin{cases} 0 & \text{if n is divisible by any square} \geq 1 \\ \text{otherwise:} & \\ 1 & \text{if n is has an even number of prime factors} \\ -1 & \text{if n is has an odd number of prime factors} \end{cases}$$

Divisibility Rules		
Given integer k expressed in base $n \ge 2$, $k = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \ldots = \sum_{i=0}^{\infty}(a_i n^i)$, $0 \le a_i < n$		
Note: $\left(\overline{a_m a_{m-1} \ldots a_0}\right)_n = \sum_{i=0}^{\infty}(a_i n^i)$, secondary subscript omission implies base10: $\overline{a_m a_{m-1} \ldots a_0} = \sum_{i=0}^{\infty}(10^i a_i)$		
Divisor (d)		**Criterion**
Basic/Specific	3, 9	If $a_0 + a_1 + a_2 + a_3 + a_4 + \ldots$ is divisible by 3 or 9
	11	If $a_0 - a_1 + a_2 - a_3 + a_4 - \ldots$ is divisible by 11
	7, 13	If $\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \overline{a_{11} a_{10} a_9} + \ldots$ is divisible by 7 or 13
	2^m, 5^m	If $\overline{a_{m-1} a_{m-2} a_{m-3} \ldots a_0}$ is divisible by 2^m or 5^m
	7	Truncate rightmost digit and subtract twice the value of said digit from the remaining integer. Repeat this process until divisibility test becomes trivial.
General	$d \mid n^m$	If $\left(\overline{a_{m-1} a_{m-2} a_{m-3} a_{m-4} \ldots a_0}\right)_n$ is divisible by d
	factor of $n^m - 1$	If $\left(\overline{a_{m-1} a_{m-2} \ldots a_1 a_0}\right)_n + \left(\overline{a_{2m-1} a_{2m-2} \ldots a_{m+1} a_m}\right)_n + \left(\overline{a_{3m-1} a_{3m-2} \ldots a_{2m+1} a_{2m}}\right)_n + \ldots$ is divisible
	factor of $n^m + 1$	If $\left(\overline{a_{m-1} a_{m-2} \ldots a_1 a_0}\right)_n - \left(\overline{a_{2m-1} a_{2m-2} \ldots a_{m+1} a_m}\right)_n + \left(\overline{a_{3m-1} a_{3m-2} \ldots a_{2m+1} a_{2m}}\right)_n - \ldots$ is divisible
	$d = xy$, $(x,y)=1$	$(x \mid k \text{ and } y \mid k) \Leftrightarrow d \mid k$
	$d \mid kn \pm 1$	Truncate rightmost digit and add $\mp k$ times the value of said digit from the remaining integer. Repeat this process until divisibility test becomes trivial.

Section D –Logarithms

For b an integer >1, $\log_b(a) = c \Leftrightarrow b^c = a$	$\log_b(b) = 1$	$\log_b(1) = 0$
$\log(a^c) = c \log(a)$	$a^{\log_a(b)} = b$	$\log\left(\frac{ab}{c}\right) = \log(a) + \log(b) - \log(c)$
$\log_a(b) \cdot \log_b(c) = \log_a(c)$	$\log_a(b) \cdot \log_b(a) = 1$	$a^{\log(b)} = b^{\log(a)}$

Section E – Analytic Geometry

Distance between line $ax + by + c = 0$ and point (x_0, y_0) in 2D plane:	Distance between the plane $ax + by + cz + d = 0$ and point (x_0, y_0, z_0) in 3D space:
$\dfrac{\lvert x_0 a + y_0 b + c \rvert}{\sqrt{a^2 + b^2}}$	$\dfrac{\lvert x_0 a + y_0 b + z_0 c + d \rvert}{\sqrt{a^2 + b^2 + c^2}}$

Section F – Inequalities

- $\mathbb{R}^+$: the set of all positive real numbers; $\mathbb{R}^-$: the set of all negative real numbers
- $a^2 + b^2 \ge 2ab$; $a^2 + b^2 + c^2 \ge ab + bc + ca$; $3(a^2 + b^2 + c^2 + d^2) \ge 2(ab + bc + cd + da + ac + bd)$
- The "quadratic-arithmetic-geometric-harmonic mean inequality:" for $a_i > 0$

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 a_3 \ldots a_n} \le \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \le \sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{n}}$$

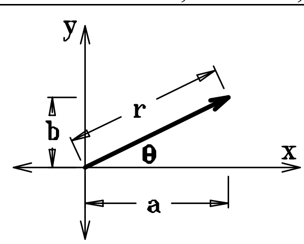
, with equalities holding iff $a_1 = a_2 = a_3 = a_4 = \ldots = a_n$.

- If constant $k>1$ and large x: $1 < k^{1/x} < x^{1/x} < \log(x) < x^{1/k} < x < x\log(x) < x^k < x^{\log(x)} < k^x < x! < x^x$

- Cauchy-Schwarz Inequality- For 2nd degree: $(a_1b_1+a_2b_2)^2 \le (a_1^2+a_2^2)(b_1^2+b_2^2)$ with equality holding iff $a_1:a_2::b_1:b_2$. In general, for any 2 sequences of real numbers, a_i and b_i, each of length n:
  $(a_1b_1+a_2b_2+a_3b_3+\ldots+a_nb_n)^2 \le (a_1^2+a_2^2+a_3^2+\ldots+a_n^2)(b_1^2+b_2^2+b_3^2+\ldots+b_n^2)$ with equality holding iff $a_1:a_2:a_3:\ldots:a_n::b_1:b_2:b_3:\ldots:b_n$.
- Chebyshev's Inequality- If $0\le a_1\le a_2\le a_3\le \ldots \le a_n$, $0\le b_1\le b_2\le b_3\le \ldots \le b_n$, then:
  $(a_1+a_2+a_3+\ldots+a_n)(b_1+b_2+b_3+\ldots+b_n) \le n\bullet(a_1b_1+a_2b_2+a_3b_3+\ldots+a_nb_n)$
- Jensen's Inequality- For a convex function f(x): $f(a_1)+f(a_2)+f(a_3)+\ldots+f(a_n) \ge n\bullet f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)$. More generally, if $b_1+b_2+\ldots+b_n=1$ and $b_i>0$, then: $b_1f(a_1)+b_2f(a_2)+b_3f(a_3)+\ldots+b_nf(a_n) \ge f(b_1a_1+b_2a_2+b_3a_3+\ldots+a_n)$

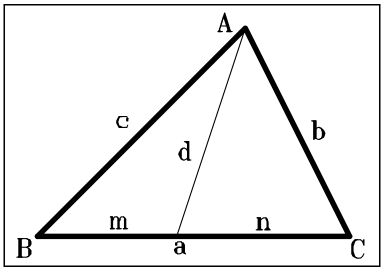
Section G – Number Systems

- $\mathbb{N}$ = natural numbers: 1, 2, 3, 4, 5, ...
- Algebraic numbers: numbers that can be solutions to polynomial equations with integer coefficients: $\sqrt{2}$, $\sqrt[5]{23}$, $\sqrt{\sqrt[5]{23}+\sqrt[7]{5}}$, ...
- Transcendental numbers: numbers that cannot be solutions to polynomials: *e*, π, ...
  - π is the ratio of the length of the circumference to the length of the diameter of a circle
  - $e = \lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^x\right)$
- if we define the square root of –1 to be *i*, then:
  - $\mathbb{C}$ = complex numbers = a+b*i*, where a,b∈R

Complex Numbers in Rectangular & Polar		$a^2+b^2=r^2$; $\tan\theta = b/a$; $a = r\bullet\cos\theta$; $b = r\bullet\sin\theta$ $Z = a+bi = r\bullet \text{cis}\,\theta$ (polar form of a complex number) The magnitude of Z, represented by $\lvert a+bi\rvert = \sqrt{a^2+b^2}$
	$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$	$\text{cis}(\theta+\gamma) = \text{cis}\,\theta\bullet\text{cis}\,\gamma$
	$(a+bi)^n = (r\,\text{cis}\,\theta)^n = r^n\bullet\text{cis}(n\theta)$	$\text{cis}(\theta-\gamma) = \frac{\text{cis}\theta}{\text{cis}\gamma}$

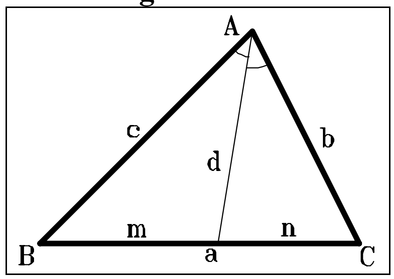
Section H – Euclidean Geometry I (The Triangle)

Stewart's Theorem



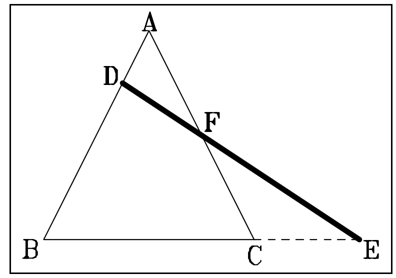
$man + dad = bmb + cnc$

Angle Bisector



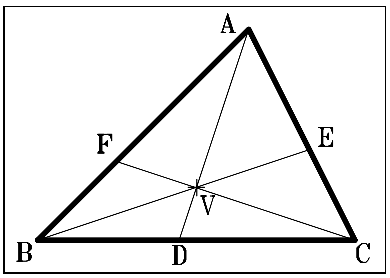
$bm = cn$; $d^2 = bc - mn$

Menelaus' Theorem



$\overline{AD}\cdot\overline{BE}\cdot\overline{CF} = \overline{DB}\cdot\overline{EC}\cdot\overline{FA}$

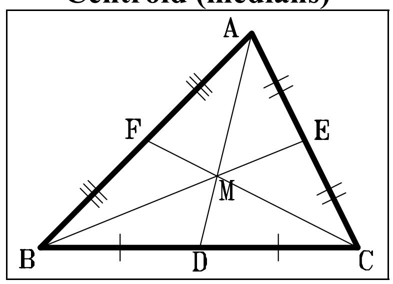
Ceva's Theorem



$\overline{AF}\cdot\overline{BD}\cdot\overline{CE} = \overline{AE}\cdot\overline{BF}\cdot\overline{CD}$

$$\frac{\overline{VD}}{\overline{AD}} + \frac{\overline{VE}}{\overline{BE}} + \frac{\overline{VF}}{\overline{CF}} = 1$$

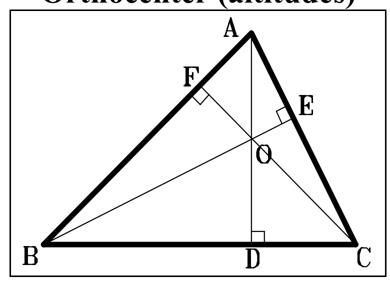
Centroid (medians)



$$\frac{\overline{AM}}{\overline{MD}} = \frac{\overline{BM}}{\overline{ME}} = \frac{\overline{CM}}{\overline{MF}} = 2$$

$K_{AFM} = K_{FBM} = K_{BDM} = K_{DCM} = K_{CEM} = K_{EAM} = \frac{1}{6}K_{ABC}$

Orthocenter (altitudes)

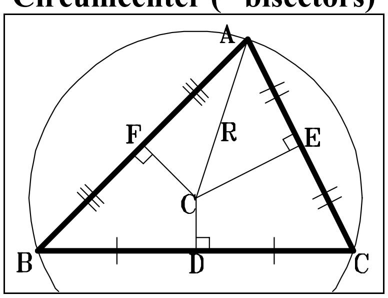


$\Delta_{AFC} \sim \Delta_{AEB} \sim \Delta_{OEC} \sim \Delta_{OFB}$

$\Delta_{BDA} \sim \Delta_{BFC} \sim \Delta_{OFA} \sim \Delta_{ODC}$

$\Delta_{CEB} \sim \Delta_{CDA} \sim \Delta_{ODB} \sim \Delta_{OEA}$

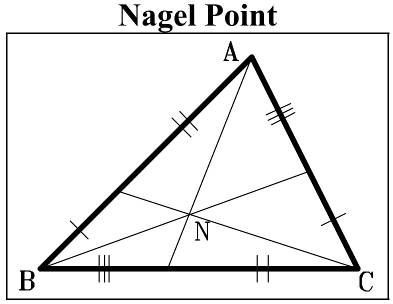
Circumcenter ($\perp$bisectors)



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

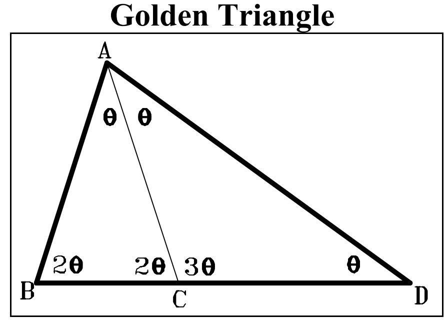
Extended Law of Sines

Nagel Point



Joins semi-perimeter points to vertices

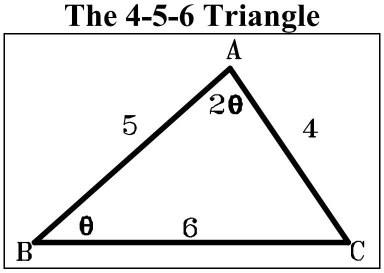
Golden Triangle



$\Delta_{ABC} \sim \Delta_{DAB}$; $\theta = 36° = \pi/5$

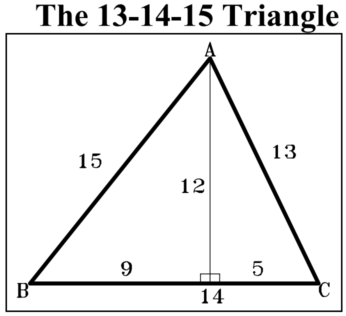
$$\frac{\overline{CD}}{\overline{BC}} = \frac{BC + CD}{CD} = \frac{\sqrt{5}+1}{2}$$

The 4-5-6 Triangle



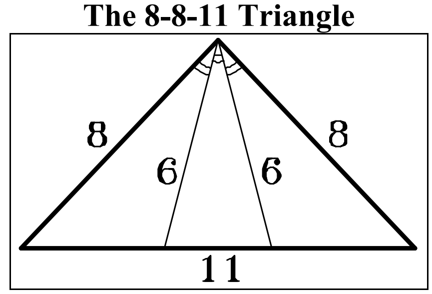
$A = 2B$; $K = \frac{15\sqrt{7}}{4}$

The 13-14-15 Triangle

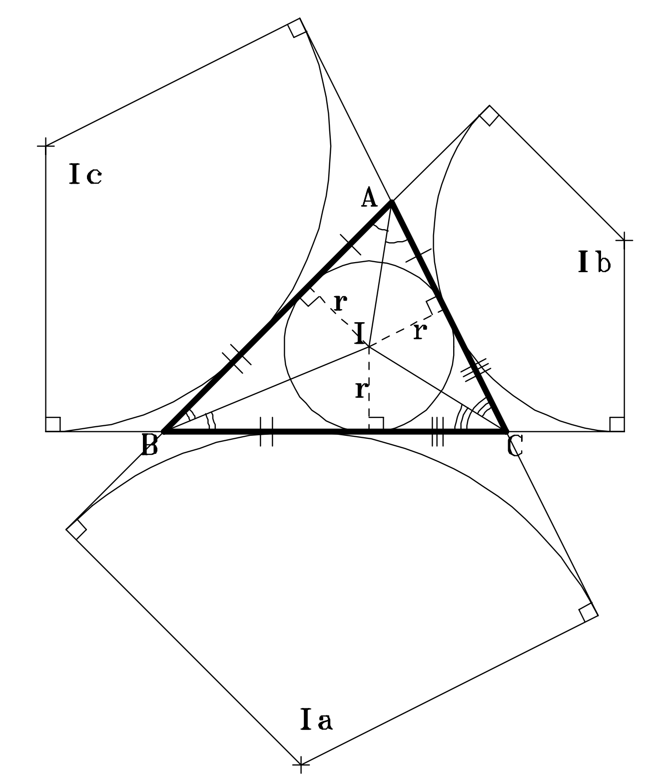


$K = 84$; $R = \frac{65}{8}$; $r = 4$

The 8-8-11 Triangle



Trisectors of the largest angle has length 6



A Triangle and Its Circles

Δ_{ABC} has sides a, b and c and angles A, B, and C.
The radius of the inscribed circle is r.
The radius of the circumscribed circle is R.
The area of the triangle is K.
The semi-perimeter of the triangle is s.
The altitude to sides a, b, c are h_a, h_b, h_c respectively.
The angle bisectors to angles A, B, C are t_a, t_b, t_c respectively.
The medians to side a, b, c are m_a, m_b, m_c respectively.
The circles tangent to each line $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and directly next to sides a, b, c are called ex-circles I_a, I_b, I_c respectively.
The radii to ex-circles I_a, I_b, I_c are r_a, r_b, r_c respectively.
The distance from I to circumcenter is d.

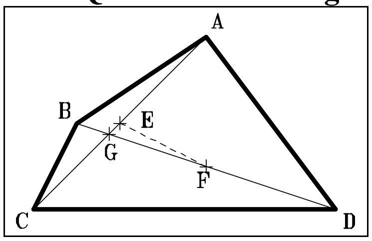
Area Formulas of the Triangle					
$K=\frac{c \cdot h_c}{2}$	$K=\frac{ab\sin C}{2}$	$K=\frac{c^2 \sin A \sin B}{2\sin C}$	$K=\frac{abc}{4R}$	K=rs	$K=\sqrt{s(s-a)(s-b)(s-c)}$
For planar triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3\ y_3)$					
$K=\frac{1}{2}\left\|\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right\|$			Coordinates of the centroid are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$		

Basic Edge Inequalities	a+b>c, b+c>a, c+a>b
Basic Angle Identities	A+B+C = 180°, $\{a,b,c\} \subset (0,\pi)$
Law of Cosines	$a^2 + b^2 = c^2 + 2ab \cos C$
Law of Tangents	tan(A)tan(B)tan(C) = tan(A)+tan(B)+tan(C)

Assorted Identities			
$r_a r_b + r_b r_c + r_c r_a = s^2$	$D^2 = R^2 - 2Rr$	$4{m_c}^2 = 2a^2 + 2b^2 + c^2$	$r_a + r_b + r_c - r = 4R$
$r = \frac{c \sin\frac{A}{2} \sin\frac{B}{2}}{\cos\frac{C}{2}}$	$r_c = \frac{K}{s-c}$	$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$	$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$
$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$	$\tan\frac{C}{2} = \frac{r}{s-c}$	$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$	$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
$t_c = \frac{2\sqrt{a \cdot b \cdot s(s-c)}}{a+b}$	$t_c = \frac{2ab\cos\frac{C}{2}}{a+b}$	$\frac{3}{4} \le \frac{m_a + m_b + m_c}{a+b+c} \le 1$	$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$

Section I – Euclidean Geometry II (The Quadrilateral)

General Quadrilateral Diagonals



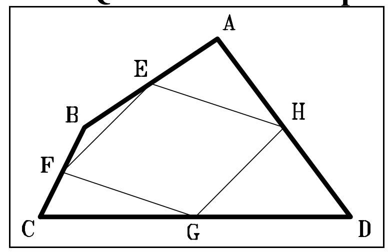
E and F are midpoints of $\overline{AC}$ and $\overline{BD}$

$K_{GAB} \bullet K_{GCD} = K_{GBC} \bullet K_{GDA}$

$K = \frac{1}{2}\overline{AC}\cdot\overline{BD}\cdot\sin\angle AGB$

$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{EF}^2$

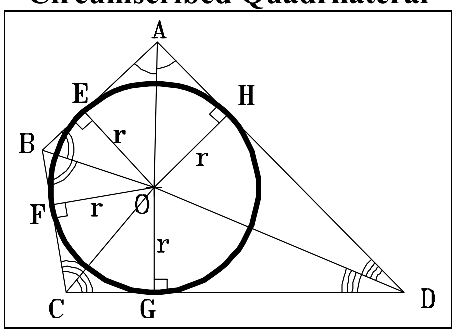
General Quadrilateral Midpoints



If $\frac{\overline{AH}}{\overline{HD}} = \frac{\overline{DG}}{\overline{GC}} = \frac{\overline{CF}}{\overline{FB}} = \frac{\overline{BE}}{\overline{EA}} = n$

Then: $\frac{K_{EFGH}}{K_{ABCD}} = \frac{n^2+1}{(n+1)^2}$

Circumscribed Quadrilateral

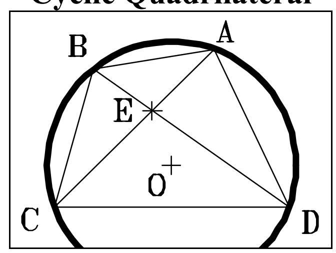


$\overline{AB} + \overline{CD} = \overline{BC} + \overline{AD} = s$; $K_{ABCD} = rs$

If $Quad_{ABCD}$ is also cyclic, then

$K = \sqrt{\overline{AB}\cdot\overline{CD}\cdot\overline{BC}\cdot\overline{AD}}$

Cyclic Quadrilateral



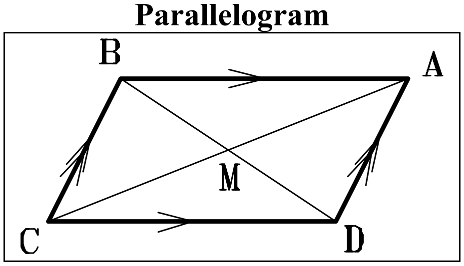
A + C = B + D = 180°

$K_{ABCD} = \sqrt{\left(s-\overline{AB}\right)\left(s-\overline{BC}\right)\left(s-\overline{CD}\right)\left(s-\overline{DA}\right)}$

$\overline{BC}\cdot\overline{AD} + \overline{AB}\cdot\overline{CD} = \overline{BD}\cdot\overline{AC}$

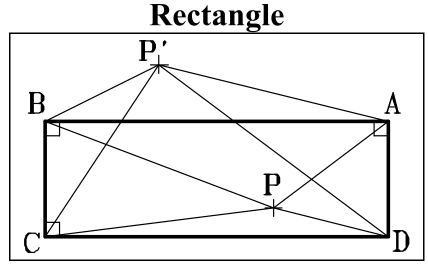
$\overline{AC}\left(\overline{BC}\cdot\overline{CD} + \overline{DA}\cdot\overline{AB}\right) = \overline{BD}\left(\overline{AB}\cdot\overline{BC} + \overline{CD}\cdot\overline{DA}\right)$

Parallelogram



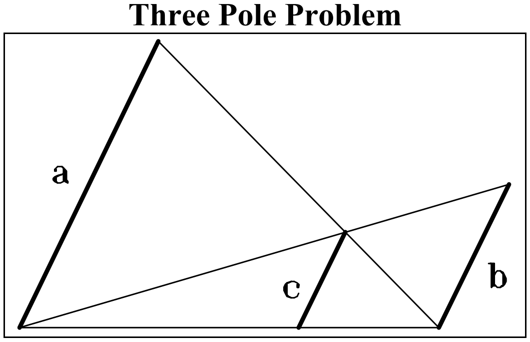
$2(\overline{BC}^2 + \overline{BA}^2) = \overline{BD}^2 + \overline{AC}^2$

Rectangle



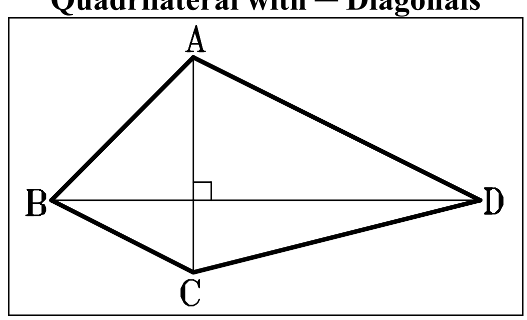
For all point P: $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$

Three Pole Problem



if a || b || c, then $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

Quadrilateral with ⊥ Diagonals



$\overline{AC} \perp \overline{BD} \Rightarrow K = \frac{1}{2}\overline{AC}\cdot\overline{BD}$

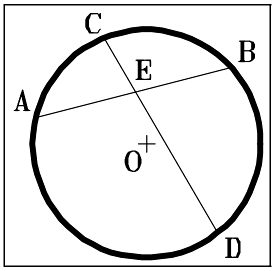
$\overline{AB}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{DA}^2$

Ptolemy's Theorem:

In any $Quad_{ABCD}$, $\overline{BD}\cdot\overline{AC} \le \overline{BC}\cdot\overline{AD} + \overline{AB}\cdot\overline{CD}$, with equality holding iff $Quad_{ABCD}$ is cyclic.

Section J – Euclidean Geometry III (The Circle)

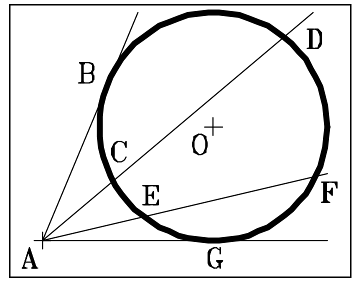
Circles



$\angle AEC = \angle BED = \frac{1}{2}\left(\overset{\frown}{AC} + \overset{\frown}{BD}\right)$

Power of the point: $\overline{AE} \cdot \overline{BE} = \overline{CE} \cdot \overline{DE}$

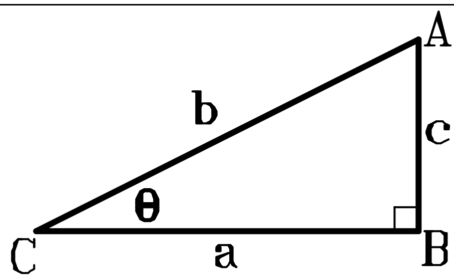
Circles 2



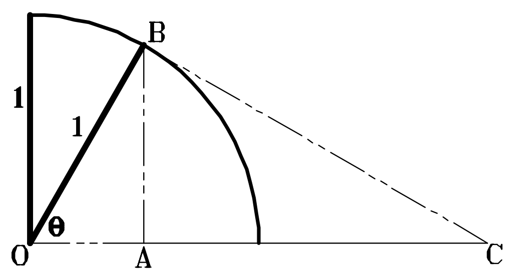
$\overline{AB} = \overline{AG}$; $\angle DAF = \frac{1}{2}\left(\overset{\frown}{DF} - \overset{\frown}{CE}\right)$

$\overline{AB}^2 = \overline{AD} \cdot \overline{AC} = \overline{AF} \cdot \overline{AE}$

Section K – Trigonometry



$\sin\theta = c/b$; $\cos\theta = a/b$; $\tan\theta = c/a$



$\sin\theta = \overline{AB}$; $\cos\theta = \overline{OA}$; $\tan\theta = \overline{BC}$

θ	15°	18°	30°	36°	45°	54°	60°	75°
sin θ	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$1/2$	$\frac{\sqrt{2(5-\sqrt{5})}}{4}$	$\sqrt{2}/2$	$\frac{\sqrt{5}+1}{4}$	$\sqrt{3}/2$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
cos θ	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{2(5+\sqrt{5})}}{4}$	$\sqrt{3}/2$	$\frac{\sqrt{5}+1}{4}$	$\sqrt{2}/2$	$\frac{\sqrt{2(5-\sqrt{5})}}{4}$	$1/2$	$\frac{\sqrt{6}-\sqrt{2}}{4}$
tan θ	$2-\sqrt{3}$	$\frac{(\sqrt{5}-1)\sqrt{2}}{2\sqrt{5+\sqrt{5}}}$	$\sqrt{3}/3$	$\frac{\sqrt{2(5-\sqrt{5})}}{\sqrt{5}+1}$	1	$\frac{(\sqrt{5}+1)\sqrt{2}}{2\sqrt{5-\sqrt{5}}}$	$\sqrt{3}$	$2+\sqrt{3}$

Pythagorean	Odd-Even Functions	Summation of Angles
$\sin^2\theta + \cos^2\theta = 1$	$\sin(-\theta) = -\sin(\theta)$	$\sin(\theta \pm \gamma) = \sin(\theta)\cos(\gamma) \pm \cos(\theta)\sin(\gamma)$
$1 + \tan^2\theta = \sec^2\theta$	$\cos(-\theta) = \cos(\theta)$	$\cos(\theta \pm \gamma) = \cos(\theta)\cos(\gamma) \mp \sin(\theta)\sin(\gamma)$
$1 + \cot^2\theta = \csc^2\theta$	$\tan(-\theta) = -\tan(\theta)$	$\tan(\theta \pm \gamma) = \frac{\tan(\theta) \pm \tan(\gamma)}{1 \mp \tan(\theta)\tan(\gamma)}$

Multiple Angles			
	$\sin 2\theta = 2\sin\theta\cos\theta$	$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	$\sin 4\theta = 4\cdot\sin\theta\cdot\cos\theta\cdot(\cos^3\theta - \sin\theta)$
	$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	$\cos 4\theta = \sin^4\theta + \cos^4\theta - 6\cos^2\theta\cdot\sin^2\theta$
	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$	$\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{\tan^4\theta - 6\tan^2\theta + 1}$

Sum to Product	Product to Sum
$\sin\theta \pm \sin\gamma = 2\sin\left(\frac{\theta\pm\gamma}{2}\right)\cos\left(\frac{\theta\mp\gamma}{2}\right)$	$\sin\theta \bullet \sin\gamma = \frac{1}{2}[\cos(\theta-\gamma) - \cos(\theta+\gamma)]$
$\cos\theta + \cos\gamma = 2\cos\left(\frac{\theta+\gamma}{2}\right)\cos\left(\frac{\theta-\gamma}{2}\right)$	$\cos\theta \bullet \cos\gamma = \frac{1}{2}[\cos(\theta-\gamma) + \cos(\theta+\gamma)]$
$\cos\theta - \cos\gamma = -2\sin\left(\frac{\theta+\gamma}{2}\right)\sin\left(\frac{\theta-\gamma}{2}\right)$	$\sin\theta \bullet \cos\gamma = \frac{1}{2}[\sin(\theta-\gamma) + \sin(\theta+\gamma)]$
$\tan\theta \pm \tan\gamma = \frac{\sin(\theta\pm\gamma)}{\cos\theta \;\square\; \cos\gamma}$	$\tan\theta \bullet \tan\gamma = \frac{\cos(\theta-\gamma)-\cos(\theta+\gamma)}{\cos(\theta-\gamma)+\cos(\theta+\gamma)}$

Square Identities	Cube Identities	½ Angle Identities	tan (θ/2) Identities
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$	$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$	$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$	$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$	$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$	$\tan^3\theta = \frac{3\sin\theta - \sin 3\theta}{3\cos\theta + \cos 3\theta}$	$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$	

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Programs Used:

Math Type 4, 5
CadKey 5
Geometer's Sketchpad 3, 4
Microsoft Word XP
Mathematica 4.1

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Bronx Science High School – Formula Sheets, Math Bulletin