Math Competition Preparation... #2

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1. **Rules & Interesting Techniques**

   (a) **Arithmetic Series** An arithmetic series is a series of numbers with a fixed difference between successive terms.

$$\begin{array}{rccccccccccc} \text{Example:} & 0 & , & 5 & , & 10 & , & 15 & , & 20 & , & ... \\ \text{Generalized Form:} & a & , & a+d & , & a+2d & , & a+3d & , & a+4d & , & ... \end{array}$$

   One can clearly see from the generalized form of the arithmetic series that the $N-$th term of a series with fixed difference d is $a+(N-1)d$. The more interesting derivation would be to find the sum of the arithmetic series:

$$\begin{array}{rcccccccc} S & = & a & + & (a+d) & + & ... & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & ... & + & a \end{array}$$

   One can see that by summing these two identical series, we would get N identical terms.

$$\begin{aligned} 2S &= (2a+(n-1)d)+(2a+(n-1)d)+...+(2a+(n-1)d) \\ 2S &= N(2a+(n-1)d) \\ S &= \frac{N(2a+(n-1)d)}{2} \end{aligned}$$

   (b) **Geometric Series** A geometric series is a series of numbers with a fixed ratio between succesive terms.

$$\begin{array}{rccccccccccc} \text{Example:} & 1 & , & 3 & , & 9 & , & 27 & , & 81 & , & ... \\ \text{Generalized Form:} & a & , & ar & , & ar^2 & , & ar^3 & , & ar^4 & , & ... \end{array}$$

   Similar to the arithmetic series case, the $N-$th term of a geometric series with fixed ratio r is ar^{n-1}. The derivation for the sum of a geometric series is also particularly interesting:

$$\begin{array}{rcccccccccccccccc} S & = & a & + & ar & + & ar^2 & + & ar^3 & + & ar^4 & + & ... & + & ar^{n-1} & & \\ Sr & = & & & ar & + & ar^2 & + & ar^3 & + & ar^4 & + & ar^5 & + & ... & + & ar^n \end{array}$$

   Subtracting the second equation from the first, we get the following:

$$\begin{aligned} S-Sr &= a-ar^n \\ S(1-r) &= a(1-r^n) \\ S &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

(c) **AM-GM** One of the most well known inequality is the Arithmetic Mean - Geometric Mean (AM-GM) Inequality. The most basic (AM-GM) inequality states that if x and y are nonnegative real numbers, then $\frac{x+y}{2} \geq \sqrt{xy}$ with equality if and only if $x = y$. A one line proof of the AM-GM inequality for two variables is as follows:

$$\frac{x+y}{2} - \sqrt{xy} = \frac{1}{2}\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$$

The AM-GM inequality generalizes to n nonnegative numbers.

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

2. **Problems**

(a) Find the sum of all integers from 1 to 100

(b) Prove that the sum of integers from 1 to N is $\frac{N(N+1)}{2}$

(c) Assuming convergence, compute

$$\frac{2}{2+\frac{2}{2+\frac{2}{2+...}}}$$

(d) In an Arithmetic Series, the third term is 18 and the seventh term is 30; find the sum of the first 17th terms.

(e) Sum the series $\frac{1}{1-\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}, ...$ to n terms.

(f) Two men set out at the same time to walk towards each other from M and N, 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, $2\frac{1}{2}$ the second hour, 3 miles the third hour, and so in an arithmetic progression. When will the men meet?

(g) It is possible to sum a geometric series if the fixed ratio r is greater than 1. Derive and proof the formula for an infinite geometric series satisfying the above criteria.

(h) The sum of an infinite number of terms of a Geometric series is 4, and the sum of their cubes is 192; find the series.

(i) Sum $1 + 3x + 5x^2 + 7x^3 + 9x^4 + ...$ to infinity with $x < 1$.

(j) Let x be any positive number. Find the minimum value of $8x + \frac{2}{x}$. (Hint: Use AM-GM)

(k) Show that the maximum value attained by $6x - x^2$ is 9.

(l) Let $x, y > 0$ and $xy = 20$. Find the minimum value of $5x + 7y$.

(m) Let x, y, z be positive. What is the least value attained by the expression $\frac{x}{y}+\frac{3y}{z}+\frac{9z}{x}$

(n) Find the minimum value of $\frac{x^2+2}{\sqrt{x^2+1}}$

(o) Show that for any integer $n > 1$,

$$\left(\frac{n+1}{2}\right)^n > n!$$

(p) Show that

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \geq n^2$$

(q) Find the minimum value of $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}$ where $x, y, z \geq 0$.

(r) *(It is highly recommended you solve the previous two problems before attempting this one.)* Let $a_i > 0$, and $S = a_1 + a_2 + ... + a_n$. Show that

$$\frac{S}{S-a_1} + \frac{S}{S-a_2} + ... + \frac{S}{S-a_n} \geq \frac{n^2}{n-1}$$