Math Competition Preparation... #4

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1. **Rules & Interesting Techniques**

   (a) **Combinatorics** A very common type of problem in math team is to find out the number of ways one can arrange N items. In order to do a deep dive into this topic, some terms have to be clearly defined.

   i. A **set** is a collection of *unordered, unique* objects.
   ii. A **multiset** is a collection of *unordered* objects.
   iii. The items of a set(multiset) are called **elements** of the set(multiset).
   iv. A set is said to have **size** N if it contains N elements.
   v. Set A is a **subset** of set B if all members of set A exists in set B.
   vi. **Permutation** illustrates the number of ways to arrange elements of a set in a definite order. The number of permutations of n elements taken r at a time $[P(n,r)]$ is

   $$P(n,r) = \frac{n!}{(n-r)!}$$

   vii. **Combination** illustrates the number of ways to arrange elements of a set without a definite order. The number of combinations of n elements taken r at a time $[C(n,r)]$ is

   $$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

   The best way to clarify the terms used above is through an example. Let $U = \{a,b,c\}, r = 2$. In this case:

   i. There are nine **permutations with repetitions**: $aa, ab, ac, ba, bb, bc, ca, cb, cc$
   ii. There are six **permutations without repetitions**: ab, ac, ba, bc, ca, cb
   iii. There are six **combinations with repetitions**: aa, ab, ac, bb, bc, cc
   iv. There are three **combinations without repetitions**: ab, ac, bc

   (b) **Inclusion-Exclusion Principle** In the set $S = 1, 2, ..., 100$, one out of every six numbers is a multiple of 6, and therefore S contains 16 multiples of 6. Similarly, S contains 14 multiples of 7. However, it is important to note that S does not contain $16 + 14$ numbers that are divisible by 6 or 7. The reason is that the sum $14 + 16$ counts the numbers that are divisible by *both* 6 and 7 twice.

   The above is a very simple illustration of what could go wrong if the counting principle is applied incorrectly. The Inclusion-Exclusion principle tells us how to keep track of what to add and what to subtract in problems like above(the below definition is borrowed from Cornell University):

Let S be a finite set, and suppose there is a list of r properties that every elemtn of S may or may not have. We call S_1 the subset of elements of S that have property 1; $S_{1,2}$ the subset of elements in S that have properties 1 and 2, etc.

Notice that $\cup S_i$ is the subset of elements of S that have at least one of the r properties. To count these elements, we

- add the number of elements that have at least one property:

$$|S_1| + |S_2| + ... + |S_r|$$

- subtract the number of elenets that have at least two properties:

$$-|S_{1,2}| - |S_{1,3}| - ... - |S_{r-1,r}|$$

- add the nubmer of elements that have at least three properties:

$$+|S_{1,2,3}| + |S_{1,2,4}| + ... + |S_{r-2,r-1,r}|$$

- ...
- add/subtract the number of elements that have all the properties:

$$\pm|S_{1,2,...,r}|$$

2. **Problems**

(a) Two dice (with six faces each) are cast. In how many different ways can they be cast such that both show even numbers or both show odd numbers.

(b) Four people deposit their hats in the cloakroom. Assuming that the hats are returned at random, find the probability that exactly k persons will get their hats back. Consider all the values of $k (0 \leq k \leq 4)$.

(c) How many 5 digit numbers have digits in strictly decreasing order? (For example, 97432 is strictly decreasing, but 97442 is not.)

(d) How many 5 digit numbers have digits in strictly increasing order?

(e) How many ways can you split 10 people into 5 pairs?

(f) A *derangement* of $(1, 2, ..., n)$ is a permutation that moves every number away from its correct position. For example $(2, 5, 4, 1, 3)$ is a derangement, but $(2, 5, 3, 1, 4)$ is not. How many derangements of $(1, 2, ..., n)$ are there?

(g) In how many ways can the number 10 be expressed as the sum of 3 non-negative integers? (Hint: Visualize the number 10 as a sum $(1 + 1 + ... + 1)$)

(h) In how many ways can the number 10 be expressed as the sum of integers greater or equal to 1.

(i) In how many ways can the number n be expressed as the sum of 3 non-negative integers?

(j) In how many ways can the number n be expressed as the sum of k non-negative integers?

(k) Find the *total* number of factors of 18,054. (Hint: Any number can be decomposed into a product of unique primes, $p_1^{a_1} \cdot ... \cdot p_n^{a_n}$. (i.e. $36 = 2^2 \cdot 3^2$)).

(l) A deck containing $4n$ $(n \geq 5)$ cards has n cards each of four different suits, numbered $1, 2, ..., n$. In how many different ways can five cards be chosen, so that they contain:

   i. five consecutive cards of the same suit?
   ii. four of the five cards with the same value?
   iii. three cards with one value and two cards with some other value?
   iv. five cards of the same suit?
   v. five successively numbered cards?
   vi. three of the five cards with the same value?
   vii. not more than two cards of the same suit?

(m) A rectangular city is divided by streets into squares. There are n such squares from north to south, and k squares from east to west. Find the number of shortest walks from the north-eastern end of the city to the south-western end.

(n) Find the number of rational terms in the expression $\left(\sqrt[3]{6}+\sqrt[4]{2}\right)^{100}$

(o) **Non-trivial** Married Couples Problem. In how many different ways can n married couples be seated around a round table so that men and women occupy alternate positions, and none of the couples is seated side by side?